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ABSTRACT

Presented are four interpretive reports and a list of current publications of the Mathematics Work Group at the Wisconsin R&D Center. The first interpretive report gives a brief overview of their work. Reports two through four are concerned with individual interviews, carried out during a longitudinal study, in which primary children are asked to solve a variety of verbal addition and subtraction problems. Report two considers correctness of answers along with some interpretations and implications of data collected thus far. Reports three and four consider addition and subtraction, respectively. The strategies and processes used to solve the problems are discussed and the findings to date are listed. (MK)

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Interpretive Report No. 1

January 1980

A Brief Introduction to the Mathematics Work Group at the Wisconsin R & D Center

by

Jim Moser

This is the first in what we hope will be a series of research reports issued by the Mathematics Work Group of the Wisconsin Research and Development Center for Individualized Schooling. It is our intent that each report will be brief, to the point, and free of excessive technical language and long data tables. We hope that the information will be of interest as well as practical value to teachers, supervisors, and curriculum developers. At the same time, we want to communicate our findings to other researchers so that they can keep abreast of what is going on in the field.

In this first report, we do not intend to present any findings nor make any interpretations of results. Rather, we want to describe in some detail the research program we are presently carrying out. Before doing that, let us tell you about the Wisconsin R & D Center. In past years, the Center was known as the Center for Cognitive Learning and was responsible for the conceptualization and development of Individually Guided Education (IGE). Other important products from the Center are the Wisconsin Design for Reading Skills Development, the Prereading Skills Program, and from our own group, the Developing Mathematical Processes (DMP) program. In the future, the Center expects a greater concentration on basic research. You should know that the Center is one of several such research and development centers located at various universities around the country (e.g., Pittsburgh, UCLA, Texas at Austin, Oregon), all of which are supported financially by the National Institute of Education (NIE).

At the present time, the principal area of interest for the Mathematics Work Group is in mathematical learning by young children of primary school age. Even this is a very large area and so we have delimited our study to the content domain of addition and subtraction. We are studying the processes that children use to solve simple verbal addition and subtraction problems and we are attempting to identify how these processes evolve as the children mature. We believe that this investigation will not only help us to better understand children's problem solving skills but will also help us to understand how youngsters acquire basic addition and subtraction concepts and skills.

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In order to study the evolution and development of problem solving skills over time, we are conducting a three-year longitudinal study. It began in fall 1978 with about 150 first-grade children from eight classes in three different schools in the Madison, Wisconsin area. After the first year one school dropped out of the study. Thus, at this time, we have just about 100 second-graders from two schools, one with four classrooms and the other with two. In addition to this major study, we will also carry out a number of smaller, short term studies. Within each of these investigations we are looking at three factors that we feel are the major determinants of children's performance on verbal problems. One is the structure or nature of the problem being posed to the child. The second is the characteristics of the child, and the third is the nature of the instruction the child has received.

The data for our studies are being gathered in three distinct ways:

1. Individual interviews. Members of our staff go into the cooperating schools and carry out individual interviews with children in some quiet location separated from the classroom. An interview session lasts about 20 minutes. For the most part the interviews have been devoted to assessing children's behavior on the verbal addition and subtraction problems although we have also used this technique for measuring pupil characteristics on other cognitive variables such as conservation, class inclusion, and memory capacity.

2. Paper and pencil instruments. Group administered tests are given to children periodically in order to measure pupil achievement on related mathematical behaviors, such as counting skills, representation skills and acquisition of basic fact combinations. These tests also give us a further measure of behaviors associated with verbal addition and subtraction problems. These written tests help us assess the effect of instruction.

3. Classroom observation. During each period of each class that arithmetic is being taught, a trained observer is present in that class. Records of what material is being taught, pupil-teacher interactions, pupil-pupil interactions, individual work, and most importantly, amount of engaged (time on task) time of selected pupils are obtained.

In subsequent interpretive reports, we will present findings from our research to date and draw some tentative conclusions about what the findings seem to suggest.

For further information about the Wisconsin R & D Center or to be put on the mailing list to receive the quarterly R & D Center News, write or call:

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Interpretive Report No. 2 January 1980

How Well Do Young Children Solve Verbal Problems?

by

Jim Moser

A common assumption of almost all school mathematics programs is that verbal problems are difficult for children of all ages, and children must first master symbolic addition and subtraction operations before they will be able to solve even simple verbal problems. Some of our current research indicates that this assumption may be false.

This tentative conclusion is based upon findings from a three-year longitudinal study of 150 children that is presently being carried out by the Mathematics Work Group of the Wisconsin Research and Development Center for Individualized Schooling. This longitudinal study is characterized in more detail in Interpretive Report No. 1 of this series. The data to be described in this report come from individual interviews carried out with primary grade children in which they are asked to solve a variety of verbal addition and subtraction problems. While the processes and strategies employed by the children are of major importance to us, we shall save a discussion of those strategies until later reports in this series. In this report, only the correctness of answers is considered along with some interpretations and implications of the data we have collected thus far.

Who are the children we are studying?

Since we want to study large numbers of students, yet have to live within the realities of limited funds, the first set of subjects all come from schools in the Madison, Wisconsin area. The three schools and eight classes involved have children drawn from middle to upper-middle class neighborhoods with an extremely small percentage of minorities. Each school uses the Developing Mathematical Processes (DMP) program which was developed at the Wisconsin R & D Center. The longitudinal study began with first grade children whose mathematical experiences in kindergarten or pre-school we can only surmise to be somewhat typical. At the beginning of the second year of the study in fall 1979, one school dropped out of the study, reducing the number of subjects to approximately 100 second-graders.

What are the verbal problems used in the study?

A major dimension upon which verbal problems can be classified is whether or not the structure or context of the problem involves action or no action. In an action situation, there is a change-causing action over a period of time that results in an increase or decrease in the size of some described entity. In contrast, there can be a static relationship existing at some fixed point in time among some described entities. The action vs. static dimension was a major factor noted in the six verbal problems used in our study. We give examples of those problems. At each administration the nouns and verbs were sytematically changed so that children would not recognize the problem situations.

Addition problems

Joining:	Wally had 3 pennies. His father gave him 6 more pennies. How many pennies does Wally have altogether? (This is an action situation.)
Part-Part-Whole:	Carol has 4 old shirts. She also has 7 new shirts. How many shirts does Carol have altogether? (This is a static situation.)

Subtraction problems

Separating:	Marie had 7 candles. She gave 5 candles to Fran. How many candles did Marie have left? (This is an action situation.)
Comparison:	Patrick has 9 fish. His sister Jill has 14 fish. How many more fish does Jill have than Patrick? (This is a static situation.)
Joining, missing-addend:	Fred has 8 M&M's. How many more M&M's does he have to put with them so that he has 13 M&M's altogether? (This is an action situation.)
Part-Part-Whole, missing-addend:	There are 5 jars of paint. Three jars are red and the rest are blue. How many jars of blue paint are there? (This is a static situation.)

Within the problems, two of three numbers from the number triple $x + y = z$. ($x < y < z$) are given. In small number problems, $5 \leq z \leq 9$; in large number problems, $11 \leq z \leq 15$. In the two addition problems, x is given before y. In the four subtraction problems, the larger addend y is always given along with the sum z.

It is important to note that all of the above examples, except the Comparison problem, deal with situations in which the two smaller numbers (either both given as in addition, or one given and the other unknown as in subtraction) represent sets that are each subsets of a larger set that is represented by the third and largest number. In the Comparison problem, the two smaller sets are not subsets of a third set -- one of them is but not the other. This can make a difference in how a child attempts to solve that problem.

How good are children at solving these problems?

Within each interview of the longitudinal study, we read to each child the six problems in four different conditions, taking the four possible combinations of smaller numbers vs. larger numbers together with presence vs. absence of manipulative materials (small plastic cubes). The diagram below shows these four combinations along with the code names, b+, b-, c+, c- assigned to those conditions.

		Presence of cubes: with	Presence of cubes: without
Size of numbers	smaller	b+	b-
	larger	c+	c-

Thus, each interview contains a maximum of 24 tasks. The length of the interview, coupled with the relatively short attention span of young children, dictates that the interview be spread over two days with about 20 minutes each day. On the first day, the "b" tasks are given, manipulatives first followed by no manipulatives. On the second day a similar pattern was followed for the "c" tasks. If a child is confused or having great difficulty, the interview is terminated early. As a consequence, the actual number of children administered a particular task will vary, with smaller numbers of subjects being given the later, harder tasks.

As of this writing, the individual interviews have been conducted four times, October 1978, January 1979, May 1979, and September 1979. The first two took place before the children received instruction in symbolic addition and subtraction. The latter two have occurred about halfway through the initial instructional sequence dealing with formal addition and subtraction. It should be noted that the DMP curriculum being used has a strong problem

solving orientation with instruction in arithmetic dealing with writing sentences to represent a variety of problem situations, as well as with basic fact learning.

In the data table on the following page, we present results in terms of numbers of children who responded with the correct answer. To be sure the table is interpreted correctly, we isolate a single entry, the January 1979 results for the Subtraction Comparison problem at the c+ level. The entry is 61/123 (50%). This indicates that 123 children were administered the task, 61 (50%) of them getting it correct. Looking at the top of the January 1979 column, we can see that 150 children were originally involved. By the time this particular task was given, 27 children had had their interview terminated earlier because the interviewer judged the individual child was having great difficulty. We assumed that those 27 children would have answered incorrectly. However, those 27 are not used in the determination of the 50% entry. The decrease in the original number of children administered the tasks from May 1979 to September 1979 reflects the fact that one school dropped out of the study during that period.

What interpretations can be given to these results?

Certainly, the results are tentative since the study is not yet complete. Although the population from which the subjects have been drawn is somewhat restricted, we are assuming that our results are generalizable to a much wider range of students until any further studies indicate that such an assumption is untenable.

- Addition problems, in general, are easier than subtraction problems.

This is consistent with findings from other research. But the thing to note is that children at the very beginning of first grade already show a high level of performance which gets even better over time. An implication for instruction might be that since addition seems to be natural to youngsters and since they do quite well at it even with limited instruction (evidence from the October 1978 and January 1979 interviews), then perhaps we should devote much less instructional time to addition and more time to other important aspects of the mathematics curriculum, such as geometry, measurement and problem solving.

- Children do better with smaller number problems than with larger number problems.

Though not evident from the table shown here, we can report that many children used fingers along with a variety of counting procedures when solving problems. And, obviously, use of fingers is easier when all numbers in a problem are less than 10. We know that children's use of fingers is considered inappropriate by many teachers. We also agree that at a certain stage of mathematical development, it is certainly better to have a child

Number of children: responding correctly/administered (%)

Problem type	Interview administration date			
	Oct. 1978	Jan. 1979	May 1979	Sept. 1979
(original # of subjects)	(144)	(150)	(150)	(99)
Addition - Joining				
b+	96/144 (67%)	126/150 (84%)	138/150 (92%)	91/99 (92%)
b-	90/118 (72%)	113/138 (82%)	137/148 (93%)	88/98 (90%)
c+	68/100 (68%)	99/127 (78%)	125/145 (86%)	83/96 (86%)
c-	44/90 (49%)	68/111 (61%)	106/139 (76%)	71/88 (81%)
Addition - Part-Part-Whole				
b+	108/144 (75%)	123/150 (82%)	137/149 (92%)	92/99 (93%)
b-	92/118 (78%)	108/135 (80%)	137/147 (93%)	89/98 (91%)
c+	72/100 (72%)	107/124 (86%)	126/145 (87%)	71/96 (74%)
c-	41/76 (54%)	68/102 (67%)	99/137 (72%)	65/88 (74%)
Subtraction - Separating				
b+	92/144 (64%)	108/150 (72%)	134/150 (89%)	82/99 (83%)
b-	74/118 (63%)	92/138 (67%)	129/148 (87%)	80/98 (82%)
c+	60/100 (60%)	80/127 (63%)	111/145 (77%)	69/96 (72%)
c-	24/90 (27%)	31/111 (28%)	68/139 (49%)	52/88 (59%)
Subtraction - Part-Part-Whole, missing-addend				
b+	67/144 (47%)	82/150 (55%)	119/150 (79%)	75/99 (76%)
b-	50/118 (42%)	71/138 (51%)	104/148 (70%)	75/98 (77%)
c+	51/100 (51%)	68/127 (54%)	105/145 (72%)	56/96 (58%)
c-	19/90 (21%)	34/110 (31%)	64/139 (46%)	46/88 (52%)
Subtraction - Comparison				
b+	75/133 (56%)	84/147 (57%)	108/149 (72%)	73/98 (74%)
b-	61/111 (55%)	76/133 (57%)	97/147 (66%)	75/98 (77%)
c+	55/97 (57%)	61/123 (50%)	85/145 (59%)	58/96 (60%)
c-	29/61 (48%)	42/89 (47%)	58/133 (44%)	47/88 (53%)
Subtraction - Joining, missing-addend				
b+	92/128 (72%)	120/147 (82%)	133/149 (89%)	89/98 (91%)
b-	86/111 (77%)	110/133 (83%)	132/147 (90%)	85/98 (87%)
c+	56/97 (58%)	86/123 (70%)	114/144 (79%)	81/96 (84%)
c-	36/55 (65%)	59/88 (67%)	92/133 (69%)	67/88 (76%)

"know the facts" than to have to rely on finger counting. But, how about first grade children? If use of fingers helps to succeed at problem solving, then maybe it should be permitted, if not encouraged. We submit that it is preferable to allow finger counting (and success!) rather than to force a too-early rote memorization of facts.

• In subtraction, action problems are easier than static problems.

The data demonstrate that the two problems having a clearly stated action component -- the Separating and the Joining, missing-addend -- prove to be easier than the two having the static relationship -- the Comparison and the Part-Part-Whole, missing-addend. This finding seems to agree with one's intuition that action oriented problems appear to be more natural to young children. Thus, when teaching specific problem solving behaviors one should probably emphasize these problems first.

• On all problems, performance improves over time.

The data suggest that instruction does indeed have an effect on performance. In view of the fact that the DMP curriculum emphasizes problem solving and has explicit instruction in some of the problem types being used, one would expect performance to improve. But looking more closely at the data, we find that the proportional increase is pretty much the same for all problems. Because the addition and the action subtraction problems had a higher success rate to begin with, their success rate in the latest interview is still better. But the initial lower degree of success with the static subtraction problems may simply reflect nothing more than children's unfamiliarity with those problem situations. The increase in success rate may be due in large part to the increased exposure to those problems, and its concomitant familiarity with those problems. In short, opportunity to learn is a most important factor in children's achievement.

• Children can solve verbal problems.

Mindful of the fact that there are varying rates of success for different problem types and different interview conditions, the clear result is that there is no overriding indication of failure for any particular problem. Here we should be careful to limit this finding to the problems used in this study. In future studies we do plan to examine more closely other problem types. Nevertheless, we have the firm conviction that working with and solving such verbal problems does give meaning and understanding to the operations of addition and subtraction. We also firmly believe that problem solving of this type deserves a greater share of the primary curriculum than it presently receives.

Additional copies of this report may be obtained by writing to Math Work Group. Wisconsin R & D Center, 1025 West Johnson Street, Madison, WI 53706.

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Interpretive Report No. 3 January 1980

A First Look at the Solving of Verbal Addition Problems

by

Jim Moser

One of the major aims of the research program being carried out by the Mathematics Work Group of the Wisconsin Research and Development Center for Individualized Schooling is to find out how young children develop in their understanding and skills related to the arithmetic operations of addition and subtraction. Of the many ways one may choose to investigate this problem, we have elected to examine children's behavior when solving verbal problems. Our interest lies not just in whether children are able to solve such problems, but more importantly in how they go about solving them. In other words, we are studying the processes and strategies employed by beginning school age youngsters and the ways in which these processes develop and change over time. In this particular report, we take a look at addition problems. In the next report (No. 4) we look at subtraction.

Where do the data come from?

We are carrying out a three-year longitudinal study that began in fall 1978 with 150 first graders from 8 classrooms in 3 different schools in the Madison, Wisconsin area. As of fall 1979, the number of subjects was reduced to 99 due to one school electing to drop out of the study. The children are individually interviewed at regular intervals during the school year. At each interview verbal problems are presented under four conditions -- smaller numbers and presence of manipulative materials to help in solving, smaller numbers without manipulatives, larger numbers with manipulatives, and larger numbers without manipulatives. In each condition, two verbal addition problems are presented. The following are exemplars of those problems.

Joining: Wally has 3 pennies. His father gave him 6 more pennies. How many pennies does Wally have altogether?

Part-Part-Whole: Carol has 4 old shirts. She also has 7 new shirts. How Many shirts does Carol have altogether?

In the actual administrations nouns and verbs are systematically changed so that children do not become too familiar with the problems. The given and unknown numbers in the problems constitute a number triple (x, y, z)

defined by $x + y = z$, $x < y < z$. In the small-number problems, $5 \leq z \leq 9$ and in the larger-number problems $11 \leq z \leq 15$.

The Joining problem has a contextual relationship of action that causes a change over time of a particular set, in this case making it larger. In contrast, there is a static, or non-action relationship present in the Part-Part-Whole problem. In all addition verbal problems, the smaller of the two addends was always given first in the problem. Problems were read to the children, repeated as necessary.

What kinds of problem-solving strategies were observed?

The strategies or processes that children used can be divided into three distinct major categories. It would also appear that these categories are ordered in terms of levels of sophistication that suggest developmental stages. The lowest level is the use of sets of physical objects to represent problem entities together with enumeration of those sets. The second level is counting and the third is the use of unobservable mental processes. We should also mention a lower level of performance which is the total inability to solve the problems at all. Instances of this were observed in the first interview given to beginning first graders, but it virtually disappeared after that.

Modeling. The child begins the solution process by setting out physical models to represent, on a one-to-one basis, the two given sets of the problem. In the two interview conditions when manipulatives (cubes) are available, they are utilized to do the modeling. In other instances, fingers are used as models, particularly in the small-number conditions. After modeling, action such as pushing together is often performed on the models. Finally, the union of the two sets is enumerated, with the counting sequence always beginning "one, two, three, etc." Ordinarily, when the enumeration process is being carried out, the child touches the elements of the model sets one-by-one. So, essentially three separate counting or enumeration sequences are carried out -- for the first addend, for the second addend, and finally for the union of the two model sets.

Counting. At this higher level, the child appears to realize that a prior-learned sequence of counting words can be used to solve the problem. Here the counting words themselves are used to represent the given sets. In the simplest example of counting, a child begins the counting sequence with, "one, two, etc." going up to the number corresponding to the size of the first addend. Then the counting continues, but at this point a double-count process must be initiated. For example, suppose the two addends are 6 and 8. After counting up to "six," the child must then utter 8 more counting words, beginning with "seven, eight, etc." and ending (if executed correctly) at "fourteen." During this latter portion, the child has to keep track of the number of counting words uttered since "six" so that he/she can stop when the eighth counting word after "six" has been uttered. The double count, or tracking, can be effected by using cubes or fingers, setting them out one at a time in synchrony with the counting words as they are spoken. Other children keep track in their heads with no overt behavior. This type of counting is called Counting All.

Many children realize that the sequence of counting words does not have to be initiated at "one." If the sequence is begun with the first, smaller number, or its successor ("six" or "seven" as in our example), then it is called Counting On from First. When a child begins with the given second, the larger number, such as in the same example cited, "nine, ten, ... fourteen," this is called Counting On from Larger. We consider this to be a more sophisticated counting behavior since it suggests the possibility that the child has some innate sense about the commutativity of addition. Certainly it is a more efficient procedure since it is quicker and fewer counting words have to be kept track of.

Mental. These higher order processes involve a mental manipulation, not of counting words and counting sequences, but rather of number concepts themselves. Within this category is use of remembered basic number facts. The other is what we have called a heuristic strategy. A heuristic is use of some known fact in order to derive some other fact. As an example, suppose the addition problem is solvable by the addition of 3 and 8. The child might say that the answer of 11 was obtained by knowing "that 3 plus 7 is 10, but 8 is one more than 7 so the answer is one more than 10 and that's 11."

What are the findings to date?

The results described in this report come from individual interviews administered in October 1978, January 1979, May 1979, and September 1979. A portion of the results were presented in an earlier report (No. 2) in this series. Because the longitudinal study is still being carried out, it is best to think of the results as tentative, subject to further interpretation as more results are obtained.

- Young children are highly successful at solving the verbal addition problems used in this study.

Right from the very beginning, the success rate was quite high. About 2 out of 3 beginning first-graders were able to correctly respond to the problems posed, when the numbers were small and roughly 1 out of 3 when larger numbers were used. This suggests the children were able to comprehend the problems and go about solving them. As time progressed, the success rate climbed to over 90% for the smaller number problems and about 89% for the larger number problems. Even here, the processes used suggested that almost all subjects understood the problems and the errors were mainly due to careless counting errors rather than a lack of understanding.

- There is no difference in performance between action and static problems.

The patterns of responses for the Joining and Part-Part-Whole problems are almost identical. This suggests that children see little or no difference between the two in terms of recognizing them as addition problems. This finding would indicate that it is not necessary to restrict the introduction of the addition concept to just joining problems as is typically done in most programs of instruction. Because static problems occur in real life, especially in subtraction, it seems wise to help children become familiar with such problems by using them freely in addition contexts where they pose no difficulty to the children.

•Counting is clearly the most frequently used strategy.

Other researchers such as Gelman and Ginsburg have shown that young children come to school with rather well developed counting ability. Our results show that children seem to be able to apply these abilities to solving verbal addition problems. Beginning first-graders almost universally employed a counting strategy of some type, with most relying on the use of models and the subsequent enumeration of the union set. As the year progressed a very sizeable number progressed to use of the more sophisticated Counting On strategies which is really quite remarkable when one considers that these counting on strategies are not generally taught by first grade teachers. Other research (e.g., Groen and Resnick) has identified the same phenomenon. Because counting is so popular and so natural, we would suggest that it would be wise for instruction to build upon this pupil strength rather than to try to suppress it. We suspect that some children may use it for most of their school lives, at least in some fashion. Some become extremely proficient and rapid in their counting ability. On the other hand, as the next finding suggests, other children divorce themselves naturally and systematically from counting as they become more skilled and comfortable with use of basic facts. In short, why try to rush children out of counting when they may either need to count or will grow out of counting on their own accord?

•There is a gradual move toward use of basic facts.

As the first year moved on and children came to the end of first grade and the beginning of second grade, more and more of them used basic facts to solve the verbal problems, particularly with the small number problems. By May 1979 in the small number problems where no cubes were present to lure children into using those manipulatives, close to half the children used basic facts correctly. Interestingly, over the summer when no formal practice or instruction was provided, the percentage of correct use of basic facts rose somewhat both with and without the presence of cubes, in the small number problems. While still not near a majority, the use of basic fact combinations with the larger number problems rose continually over the course of the year. Obviously, instruction is taking effect. Paper and pencil group-administered tests given to the children confirm that progress in mastery of basic facts is taking place. Our interview data suggest that children are able to apply this knowledge to the solution of verbal problems. We would claim that the verbal problems give more meaning to addition and help the children see why it is useful to learn those facts, not in isolation, but as a problem solving tool.

Additional copies of this report may be obtained by writing to Math Work Group Wisconsin R & D Center, 1025 West Johnson Street, Madison, WI 53706.

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Interpretive Report No. 4 January 1980

A First Look at the Solving of Verbal Subtraction Problems

by

Jim Moser

One of the major aims of the research program being carried out by the Mathematics Work Group of the Wisconsin Research and Development Center for Individualized Schooling is to try to find out how young children develop in their understanding and skills related to the arithmetic operations of addition and subtraction. Of the many ways one may choose to investigate this problem, we have elected to examine children's behavior when solving verbal problems. Our interest lies not just in whether children are able to solve such problems, but more importantly in how they go about solving them. In other words, we are studying the processes and strategies employed by beginning school age youngsters and the ways in which these processes develop and change over time. In this particular report, we look at subtraction problems. In the previous report (No. 3) we examined addition problems.

Where do the data come from?

We are carrying out a three-year longitudinal study that began in Fall 1978 with 150 first graders from 8 classrooms in 3 different schools in the Madison, Wisconsin area. As of Fall 1979, the number of subjects was reduced to 99 due to one school electing to drop out of the study. The children are interviewed at regular intervals during the school year. Verbal problems are presented under four conditions--smaller numbers and presence of manipulative materials, smaller numbers without manipulatives, larger numbers with manipulatives, and larger numbers without manipulatives. In each condition, four verbal subtraction problems are presented. The following are exemplars of those problems.

Separating:	Marie had 7 candles. She gave 5 candles to Fran. How many candles did Marie have left?
Comparison:	Patrick has 9 fish. His sister Jill has 14 fish. How many more fish does Jill have than Patrick?
Joining, missing-addend:	Fred has 8 M&M's. How many more M&M's does he have to put with them so he has 13 M&M's altogether?
Part-Part-Whole, missing addend:	There are 5 jars of paint. Three jars are red and the rest are blue. How many jars of blue paint are there?

Note that the Separating and the Joining, missing-addend problems have a contextual relationship of action that causes a change over time in the size of some particular

set. In the former problem, the size is decreased while in the latter the size is increased. In contrast, there is a static, or non-action relationship present in the Comparison and the Part-Part-Whole, missing-addend problems. In all problems the two given numbers and the unknown number (solution) constitute a number triple (x, y, z) defined by $x + y = z$ and $x < y < z$. In the small number problems $5 \leq z \leq 9$ and in the larger number problems $11 \leq z \leq 15$. For these subtraction situations, the larger addend (y) and the sum (z) were always the number given. All problems were read to the children, repeated as necessary.

What kinds of problem-solving strategies were observed?

The strategies or processes that children used can be divided into three distinct major categories. It would also appear that these categories are ordered in terms of levels of sophistication that suggest developmental stages. The lowest level is the use of sets of physical objects to represent problem entities together with enumeration of those sets. The second level is counting and the third is the use of unobservable mental processes. We should also mention a lower level of performance which is the total inability to solve the problems at all. Instances of this were observed in the first interview given to beginning first graders, but it virtually disappeared after that.

Modeling. The child begins the solution process by setting out physical models to represent, on a one-to-one basis, one or both of the given sets in the problem. In the two interview conditions when manipulatives (cubes) were available, they were utilized to do the modeling. In other instances, fingers were used as models, particularly in the small-number condition. After modeling, some action was performed. We observed three very clearly distinct types of actions on physical models which correspond directly to the three major types of subtraction situations - Subtractive, Additive, and Comparative.

Separating From. Suppose the numbers were 6 and 9. The child begins by setting out nine cubes or nine fingers. Then, one-by-one, six of those objects are removed, with the child counting "one, two, ... six" as they are removed. Counting the remainder set of 3 gives the solution.

Adding On. Again, suppose the numbers were 6 and 9. The child begins by setting out six cubes or six fingers. Then, one-by-one, objects are adjoined to the set of 6 until a new, single set of 9 is formed. Enumerating how many objects were added on gives the solution.

Matching. Again, let's use the numbers 6 and 9. The child begins by setting out two sets of objects, one with 6 objects and the other with 9. (This is virtually impossible to do with fingers, except when both numbers in the problem are less than 5.) Then the child puts the two sets into one-to-one correspondence by a physical matching or lining up of the two sets. Counting the excess of the larger over the smaller gives the solution.

Counting. At this higher level, the child appears to realize that a prior-learned sequence of counting words can be used to solve the problem. Here the counting words themselves are used to represent the given sets. Both backward and forward counting were observed. Since the counting strategies observed in solution of subtraction problems never involved the initiation of the counting sequence withe the number "one" it was necessary for the child to employ a double-count procedure of some kind to keep track of the number of counting words uttered.

Counting Down From. Let's stay with our example of 6 and 9. A backwards counting sequence is begun. Some children say "nine, eight , seven, etc." while others say "eight, seven, six, etc." In either case, the child uses a tracking device to end the sequence when six counting words have been spoken, and determines the solution to be 3. This is a counting analogue to the Separating From strategy described in the above section.

Counting Down To. Although occurring very rarely, this strategy deserves mention. Using 6 and 9 as the numbers for an example, the child begins the backward counting sequence with "nine" or with "eight" and continues until "six." Although four words, "nine, eight, seven, six" have been spoken in the first case, the child somehow knows that either "nine" or "six" should not be counted and decides that the solution is 3. In the second case, only three counting words have been spoken and the solution is again 3.

Counting Up from Given. Here a forward counting sequence is begun with the successor to 6, namely "seven" and continued to "eight, nine." Since 3 counting words have been spoken, the solution is known to be 3. This is the counting analogue to the Adding On strategy described above.

Mental. These higher order processes involve a mental manipulation, not of counting words and sequences, but rather of number concepts themselves. Within this category is use of remembered number facts. The other is what we have called a heuristic strategy. A heuristic is use of some known fact in order to derive some other fact. As an example, let's use one last time our problem involving 6 and 9. The child might say something to the effect, "I know that 6 and four makes ten. Since this problem has 9 which is one less than ten, the answer must be one less than four which is 3."

What are the findings to date?

Resarch and classroom experience over the years have shown that subtraction has tended to be more difficult than addition for school age children. The overall results of our investigation into the solution of subtraction verbal problems are consistent with previous findings. But analysis of the complex structure underlying the abstract operation of subtraction would predict such results. The verbal addition problems used in our study appear to have the same structure as interpreted by children's behavior in solving those problems (See Interpretive Report No. 3 in this series) regardless of whether the problem is an action-oriented Joining problem or a static Part-Part-Whole problem. Such is not the case with subtraction. There are three identifiably different structures underlying subtraction verbal problems -- subtractive, additive, and comparative. Our analysis of results to data reflects this complexity of subtraction.

- Problem structure in subtraction verbal problems appears to be the major factor in determining how a young child attempts to solve a given problem.

Children in the early stages of instruction do not use a single strategy to solve all problems. Rather, they are quite flexible and adopt differing strategies, depending upon the problem presented to them. But the strategy chosen reflected the structure of the problem. For the Separating problem, the overwhelming choice was the Separating From strategy, or when manipulatives were not readily available, its counting analogue, Counting Down From. For the Comparison problem, the favorite strategy was Matching when the manipulatives were available. (As noted earlier, Matching is extremely difficult without manipulatives.) For the Joining, missing-addend, its additive structure was reflected by the very high incidence of use of the additive strategies, Adding On

and Counting Up from Given. The structure of the Part-Part-Whole, missing-addend problem is not as pronounced as that of the other three problems. Consequently, the choice of strategy tended to be a bit ambiguous although most children used a separating strategy.

- Children are remarkably adaptive in their ability to use a wide variety of strategies.

This result is related to the previous finding but is an extension of it. Here we are suggesting that even within the same problem structure children can show ingenuity depending upon the condition of the problem presentation. As an example, let's look at the Comparison problem. As we noted above, the Matching strategy was chosen by many children because that strategy most closely reflected the structure of the problem. However, when the two conditions of no manipulatives existed, the children were essentially rendered unable to employ the Matching strategy. Rather than give up, they turned to alternative strategies. One was use of fingers (when the numbers were small) as models for either the Separating From or the Adding On strategies. When the numbers were larger and finger modeling became very difficult, they turned to counting strategies. Actually, this latter phenomenon occurred on all four problems in the last condition--larger numbers and no manipulatives present. Since direct modeling, either by cubes or by fingers, became tedious, more instances of use of counting procedures were observed. All of this suggests that children have a diverse repertoire of solution strategies and that they are quite selective in their choice from this repertoire, using whatever the problem structure and the interview condition dictates.

- There is a strong tendency to use direct modeling strategies.

As we just indicated, direct modeling is difficult in the fourth interview condition. However, in the other three children opted to choose modeling with fingers or cubes. This tendency exhibited itself more noticeably in the earlier interviews when the children were younger and less comfortable with the problems. We have a sense that children are innately more self-confident with addition than with subtraction as evidenced by a greater incidence of use of the more primitive types of strategy with subtraction than with addition. They know more basic addition facts than they do subtraction facts. Also, counting back is more difficult than counting forward. Some of the problems suggested a backwards counting strategy. If counting were employed, the children may have realized that their own backward counting was less proficient than their forward counting. Thus, they may have decided to go with the simpler and more reliable modeling procedures that require forward counting.

- The action problems are much less ambiguous than the others.

A related result was reported in Interpretive Report No. 2 of this series where we looked at correctness of responses. There we saw that the two action problems--the Separating and the Joining, missing-addend--had a higher success rate. Here we can report that the choice of strategy was very high in terms of a single strategy, Separating or Counting Down From for the Separating problem and Adding On or Counting Up from Given for the Joining, missing-addend. If you go back and read those problems, you can see how direct and clear those two are. The ambiguity of the Comparison and the Part-Part-Whole is reflected by the variety of strategies employed by children on those two. Even though many used Matching for the Comparison, there was a high incidence of other strategies, even when manipulatives were present.

What are the implications for instruction?

Subtraction is more complex than addition. As a result, children do not do quite as well on it as they do on addition. Yet, we are not saying that children fail on

on subtraction verbal problems. In fact, they do quite well. But, one implication is that subtraction probably requires more instructional time than addition. Secondly, since our results show that complete failure is not evident at any time, one should consider providing initial subtraction activities simultaneously with addition, rather than waiting until addition is mastered. Thirdly, as with addition, the success rate with subtraction verbal problems is sufficiently high across all problem types to indicate that children are able to understand the problems and cope with them even before formal arithmetic instruction begins. Thus, there seems to be no valid reason why instruction dealing with verbal problem solving cannot be carried on directly with the commencement of first grade mathematics. There seems to be no need to wait until basic facts have been mastered.

Perhaps the most important implication for instruction deals with the underlying structure of subtraction. Looking ahead to the symbolic treatment of subtraction and formal computational algorithms, subtraction is treated exclusively as "take away" which is to say, in the subtractive sense. But this is not the only way subtraction can be viewed! Analysis of verbal problems indicates the existence of the comparative and additive interpetations as well. And what is more emphatic, our interview data indicates that children think of subtraction in these three ways. Thus, the future abstract treatment of subtraction could very well be counter-intuitive to many children, especially in the case of the missing-addend problems. This may well be a cogent reason why children have such trouble with subtraction. Formal instruction is emphasizing subtractive thinking while the children may well be thinking in an additive way. We are not so bold as to suggest a foolproof way out of this dilemma. Nevertheless, we do advance the suggestion that perhaps teachers and curriculum writers be aware of this potential conflict in children's thinking towards subtraction, and also consider formal teaching of alternative ways of thinking about problem solving.

Additional copies of this report may be obtained by writing to: Math Work Group, Wisconsin R & D Center, 1025 West Johnson Street, Madison, WI 53706.

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Date: October 1979
Price: Offset: $3.00 (25 still available) Xerox: $7.50
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Authors: Vicky L. Kouba and James M. Moser
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Date: October 1979
Price: Offset: $10.50 (25 still available) Xerox: $35.00
Description: Curriculum products used during Mathematics Work Group longitudinal study were tried out prior to use. Majority of this report is the appendix containing actual copies of all instructional materials for six instructional units that teach problem analysis, horizontal and vertical sentence writing, and acquisition of basic addition and subtraction facts.

Theoretical Papers

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Description: A characterization of the current longitudinal study being carried out by the Mathematics Work Group. Emphasis is given to results obtained from the first year of the study in which 150 first graders were individually interviewed on three separate occasions. Implications for instruction are presented.

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Description: Originally in French, this is the presentation by Professor Servais to the Fourth Inter-American Conference on Mathematical Education, Caracas, Venezuela, December, 1975.

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